

# RECURRENCE, POINTWISE ALMOST PERIODICITY AND ORBIT CLOSURE RELATION FOR FLOWS AND FOLIATIONS

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**ABSTRACT.** In this paper, we obtain a characterizations of the recurrence of a continuous vector field  $w$  of a closed connected surface  $M$  as follows. The following are equivalent: 1)  $w$  is pointwise recurrent. 2)  $w$  is pointwise almost periodic. 3)  $w$  is minimal or pointwise periodic. Moreover, if  $w$  is regular, then the following are equivalent: 1)  $w$  is pointwise recurrent. 2)  $w$  is minimal or the orbit space  $M/w$  is either  $[0, 1]$ , or  $S^1$ . 3)  $R$  is closed (where  $R := \{(x, y) \in M \times M \mid y \in \overline{O(x)}\}$  is the orbit closure relation). On the other hand, we show that the following are equivalent for a codimension one foliation  $\mathcal{F}$  on a compact manifold: 1)  $\mathcal{F}$  is pointwise almost periodic. 2)  $\mathcal{F}$  is minimal or compact. 3)  $\mathcal{F}$  is  $R$ -closed. Also we show that if a foliated space on a compact metrizable space is either minimal or both compact and without infinite holonomy, then it is  $R$ -closed.

## 1. PRELIMINARIES

In [AGW] and [H], it is showed that the following properties are equivalent for a finitely generated group  $G$  on either a compact zero-dimensional space or a graph  $X$ : 1)  $(G, X)$  is pointwise recurrent. 2)  $(G, X)$  is pointwise almost periodic. 3) The orbit closure relation  $R = \{(x, y) \in X \times X \mid y \in \overline{G(x)}\}$  is closed.

In this paper, we study the equivalence for these three notions for vector fields on surfaces and codimension one foliations on manifolds, and show the some equivalences. We assume that every quotient space has the usual quotient topology and that every decomposition consists of non empty elements. By a decomposition, we mean a family  $\mathcal{F}$  of pairwise disjoint subsets of a set  $X$  such that  $X = \sqcup \mathcal{F}$ . Let  $X$  a connected topological space and  $\mathcal{F}$  a decomposition of  $X$ . For any  $x \in X$ , denote by  $L_x$  the element of  $\mathcal{F}$  containing  $x$ . Write  $E_{\mathcal{F}} := \{(x, y) \mid y \in L_x\}$ . Then  $E_{\mathcal{F}}$  is an equivalence relation (i.e. a reflexive, symmetric and transitive relation). For a (binary) relation  $E$  on a set  $X$  (i.e. a subset of  $X \times X$ ), let  $E(x) := \{y \in X \mid (x, y) \in E\}$  for an element  $x$  of  $X$ . For any  $A \subseteq X$ , let  $E(A) := \cup_{y \in A} E(y)$ .  $A$  is said to be  $E$ -saturated if  $A = \cup_{x \in A} E(x)$ . Let  $1_X := \{(x, x) \mid x \in X\}$  the diagonal on  $X \times X$ . Thus,  $1_X \subseteq E$  if and only if  $E$  is reflexive (i.e.  $x \in E(x)$  for all  $x \in X$ ). Let  $E^{-1} := \{(y, x) \mid (x, y) \in E\}$  (i.e. the image of  $E$  under the bijection  $T : X \rightarrow X$  which interchanges coordinates). Clearly,  $E$  is symmetric if and only if  $E = E^{-1}$ . For any relation  $E$  on  $X$ ,  $E$  is transitive if and only if  $E(E(x)) \subseteq E(x)$ . For an equivalence relation  $E$ , the collection of equivalence classes  $\{E(x) \mid x \in X\}$  is a decomposition of  $X$ , denoted by  $\mathcal{F}_E$ . Note that decompositions (consisting nonempty elements) can be corresponded to equivalence relations. Therefore we can identify

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decompositions with equivalence relations. For a relation  $E$  on a topological space  $X$ ,  $E$  define the relation  $\hat{E}$  on  $X$  with  $\hat{E}(x) = \overline{E(x)}$ . Denote by  $\overline{E}$  the closure of  $E$  in  $X \times X$ . We call  $E$  pointwise almost periodic if  $\hat{E}$  is an equivalence relation,  $R$ -closed if  $\hat{E}$  is closed (i.e.  $\hat{E} = \overline{E}$ ), compact (resp. connected) if each element of  $E$  is compact (resp. connected), and it is minimal if each element of  $E$  is dense in  $X$ . By identification, we also said that  $\mathcal{F}$  is  $R$ -closed if so is  $E_{\mathcal{F}}$  and others are defined in similar ways. Notice that if  $\mathcal{F}$  is either a foliation or the set of orbits of a flow, then  $\hat{E}_{\mathcal{F}}$  is transitive. By a flow, we mean a continuous action of a topological group  $G$  on  $X$ . We call that  $\mathcal{F}$  is trivial if it consists of single points or is minimal. We characterize the transitivity for  $\hat{E}$ .

**Lemma 1.1.**  *$\hat{E}$  is transitive if and only if  $E(\hat{E}(x)) \subseteq \hat{E}(x)$ . When  $\hat{E}$  is an equivalence relation, this says that each  $\hat{E}(x)$  is a union of  $E$ -equivalence classes.*

*Proof.* Since  $\hat{E}(x)$  is closed,  $E(y) \subseteq \hat{E}(x)$  implies  $\hat{E}(y) = \overline{E(y)} \subseteq \hat{E}(x)$ .  $\square$

Now we state a useful tool.

**Lemma 1.2.** *If  $E$  is an equivalence relation, then  $\hat{E}$  is an equivalence relation if and only if it is symmetric (i.e. symmetry implies transitivity).*

*Proof.* Let  $y \in \hat{E}(x)$  and  $z \in E(y)$ . By the symmetry assumption, we have  $x \in \hat{E}(y)$ . Since  $E$  is an equivalence relation,  $E(y) = E(z)$  and so  $\hat{E}(y) = \hat{E}(z)$ . So  $x \in \hat{E}(z)$  and, by symmetry again,  $z \in \hat{E}(x)$  and so  $E(y) \subseteq \hat{E}(x)$ . Then  $E(\hat{E}(x)) \subseteq \hat{E}(x)$ . Hence  $\hat{E}$  is transitive by Lemma 1.1.  $\square$

Notice that the twist map  $T : X \rightarrow X$  is a homeomorphism and so  $E = E^{-1}$  implies  $\overline{E} = \overline{E}^{-1}$ . In particular,  $\overline{E}$  is symmetric whenever  $E$  is an equivalence relation. In particular, the previous lemma implies the following statement.

**Corollary 1.3.** *Suppose that  $E$  is an equivalence relation. If  $\hat{E}$  is closed, then  $\hat{E}$  is an equivalence relation.*

This is interpreted as the following statement.

**Corollary 1.4.** *If  $\mathcal{F}$  is an  $R$ -closed decomposition, then  $\mathcal{F}$  is pointwise almost periodic.*

The converse of this corollary is not true (see Example 4).

A map  $p : X \rightarrow Y$  is called perfect if it is continuous, closed, surjective and each fiber  $p^{-1}(y)$  for any  $y \in Y$  is compact.

**Lemma 1.5.** *If a relation  $E$  is closed and  $X$  is  $T_1$ , then  $E(x)$  is closed for any  $x \in X$ . Moreover if  $E$  is an equivalence relation and  $X$  is compact metrizable, then the quotient map  $q : X \rightarrow X/E$  is perfect.*

*Proof.* Note that each single point set is closed in a  $T_1$ -space. Since  $\{x\} \times E(x) = E \cap (\{x\} \times X)$ , we have  $E(x)$  is closed in  $X$ . Suppose that  $E$  is an equivalence relation and  $X$  is compact metrizable. We will show that  $q$  is closed. Otherwise there is a closed subset  $B$  of  $X$  such that  $E(B)$  is not closed. Fix any  $y \in \overline{E(B)} - E(B)$ . Let  $(x_n)$  be a sequence in  $B$  and  $y_n \in E(x_n)$  such that  $y_n \rightarrow y$ . Then  $(x_n, y_n) \in E$ . Since  $B$  is closed and so compact, we may assume that  $x_n$  converges to some element  $x \in B$ , by taking a subsequence of  $x_n$ . Since  $y \notin E(x) \subseteq E(B)$ , we have  $(x, y) \notin E$ . Since  $(x_n, y_n) \in E$  and they converge to  $(x, y)$ , we have  $(x, y) \in \overline{E}$ ,

which contradicts that  $E$  is closed. Since each fiber of every element of  $X/E$  is of the form  $E(x)$  for some  $x \in X$  and so compact, we have that  $q$  is perfect.  $\square$

An pointwise almost periodic  $\mathcal{F}$  is weakly almost periodic in the sense of Gottschalk [G] if the saturation  $\bigcup_{x \in A} \overline{L_x}$  of orbit closures for any closed subset  $A$  of  $X$  is closed. Note that an pointwise almost periodic decomposition  $\mathcal{F}$  is weakly almost equivalent if and only if the quotient map  $q : X \rightarrow X/\hat{E}_{\mathcal{F}} (= X/\hat{\mathcal{F}})$  is closed. This implies the following interpretation.

**Lemma 1.6.** *Suppose that  $X$  is a compact metrizable space. If a decomposition  $\mathcal{F}$  is  $R$ -closed, then  $\mathcal{F}$  is weakly almost periodic.*

## 2. GENERAL CASES FOR FLOWS AND FOLIATED SPACES

For an equivalence relation  $E$  on  $X$  and for an element  $x$  of  $X$ , define the class  $\tilde{E}(x)$  of  $x$  by  $\tilde{E}(x) := \{y \in X \mid \hat{E}(y) = \hat{E}(x)\}$ . These classes imply an equivalent relation on  $E$ . Then the quotient space by this equivalent relation is denoted by  $X/\tilde{E}$  and is called the orbit class space. Write  $\tilde{\mathcal{F}} := \mathcal{F}_{\tilde{E}}$ . By identification,  $\tilde{L} \in \tilde{\mathcal{F}}$  if and only if there is  $L \in \mathcal{F}$  such that  $\tilde{L} = \{y \in X \mid \overline{L} = \overline{L_y}\}$ . Note that this definition is an analogy for topological dynamics [BHSV]. In the case that  $\mathcal{F}$  is pointwise almost periodic,  $X/\tilde{\mathcal{F}}$  is exactly the quotient space of  $\mathcal{F}_{\tilde{E}}$  with the quotient topology and  $X/\tilde{\mathcal{F}} = X/\tilde{E}_{\mathcal{F}} = X/\hat{E}_{\mathcal{F}}$ . If  $\hat{E}$  is an equivalence relation, then  $\tilde{E} = \hat{E}$  and so  $X/\tilde{E} = X/\hat{E}$ .

**Lemma 2.1.** ([Du] XI, Theorem 5.2 (p.235)) *Let  $p : X \rightarrow Y$  be a perfect map. Then If  $X$  is either Hausdorff, metrizable, regular, or 2nd countable, then so is  $Y$ .*

This fact implies the following lemma.

**Lemma 2.2.** *Let  $E$  be an equivalence relation in  $X$ . If  $X/E$  is Hausdorff, then  $E$  is closed. If  $X$  is compact metrizable, then the converse holds.*

*Proof.* Suppose that  $X/E$  is Hausdorff. Then the diagonal of  $X/E \times X/E$  is closed and so  $E$  which is the pre-image of it is also closed. Conversely, suppose that  $X$  is compact metrizable and  $E$  is closed. By Lemma 1.5, we have that the quotient map  $q : X \rightarrow X/E$  is perfect. By Lemma 2.1, we obtain that  $X/E$  is Hausdorff.  $\square$

This implies the useful tool.

**Lemma 2.3.** *Let  $\mathcal{F}$  be a pointwise almost periodic decomposition on a compact metrizable space  $X$ . Then  $\mathcal{F}$  is  $R$ -closed if and only if  $X/\tilde{\mathcal{F}}$  is Hausdorff.*

*Proof.* Since  $\mathcal{F}$  is pointwise almost periodic, we have that  $\hat{E}_{\mathcal{F}}$  is an equivalence relation. Suppose that  $\mathcal{F}$  is  $R$ -closed. Then  $\hat{E}_{\mathcal{F}}$  is closed. By Lemma 2.2,  $X/\hat{E}_{\mathcal{F}} = X/\tilde{\mathcal{F}}$  is Hausdorff. Conversely, suppose that  $X/\tilde{\mathcal{F}}$  is Hausdorff. Since  $X/\hat{E}_{\mathcal{F}} = X/\tilde{\mathcal{F}}$ , Lemma 2.2 implies that  $\hat{E}_{\mathcal{F}}$  is closed. This shows that  $\mathcal{F}$  is  $R$ -closed.  $\square$

In this lemma, the compactness of  $X$  is necessary even if  $X$  is Hausdorff (see Example 3). However we don't know whether the condition "compact and Hausdorff" is necessary when  $\mathcal{F}$  is either the orbit space of a flow or a foliation. Now we state an observation.

**Lemma 2.4.** *Let  $\mathcal{F}$  be a decomposition on  $X$ . If either  $\mathcal{F}$  is minimal or  $X/\mathcal{F}$  is Hausdorff, then  $\mathcal{F}$  is  $R$ -closed.*

*Proof.* If  $\mathcal{F}$  is minimal, then  $R = \hat{E}_{\mathcal{F}}$  is a single point set and so closed. If  $X/\mathcal{F}$  is Hausdorff, then all orbits are closed and so  $\mathcal{F}$  is pointwise almost periodic. By Lemma 2.2, it suffices to show that  $X/\tilde{\mathcal{F}}$  is Hausdorff. But  $X/\mathcal{F} = X/\tilde{\mathcal{F}}$  is Hausdorff.  $\square$

This observation implies the following statement.

**Proposition 2.5.** *A foliated space  $(X, \mathcal{F})$  on a compact metrizable space either which is minimal or which is compact and without infinite holonomy, is  $R$ -closed.*

*Proof.* If  $\mathcal{F}$  is minimal, then Lemma 2.4 implies that  $\mathcal{F}$  is  $R$ -closed. If  $\mathcal{F}$  is a compact foliated space without infinite holonomy, then Theorem 4.2 [Ep] implies that  $X/\mathcal{F}$  is Hausdorff. By Lemma 2.4, we have that  $\mathcal{F}$  is  $R$ -closed.  $\square$

Notice that the set of  $R$ -closed foliations contains properly the set of foliations which are minimal or “compact and without infinite holonomy” [Y]. The author is interested in the characterization of the  $R$ -closedness for codimension two foliations. Our statement is similar to the statement [V](A.2)3. However notice that it is not true, because pointwise almost periodicity does not correspond to  $R$ -closedness (cf. Example 4).

### 3. ON THE $R$ -CLOSEDNESS OF $\mathcal{F}$

In this section, we will show that the following three notions are equivalent for an equivalence relation  $E$  on a compact metrizable space : 1)  $R$ -closed, 2)  $D$ -stable, and 3)  $L$ -stable. Recall that a net is a function from a directed set to a topological space. A point  $x$  in  $X$  is called  $D$ -stable (or of characteristic 0) if  $\hat{E}(x) = D(x)$  for any  $x \in X$ , where  $D(x)$  is its (bilateral) prolongation defined as follows:  $D(x) = \{y \in X \mid y_{\alpha} \in E(x_{\alpha}), y_{\alpha} \rightarrow y, \text{ and } x_{\alpha} \rightarrow x \text{ for some nets } (y_{\alpha}), (x_{\alpha}) \subseteq X\}$ . An equivalence relation  $E$  is called  $D$ -stable (or of characteristic 0) if each point is  $D$ -stable (i.e.  $D = \hat{E}$ ). Note that some authors require also non-triviality for the definition of  $D$ -stability. When  $\mathcal{F}$  is the set of orbits of a vector field  $v$ , the  $D$ -stability of  $\mathcal{F}$  is equivalent to the usual  $D$ -stability of  $v$ . When  $\mathcal{F}$  is the set of orbits of a vector field on a Hausdorff space, the following lemma is known (Theorem 1.7. [EK]).

**Lemma 3.1.** *Suppose that  $X$  is Hausdorff. Then an equivalence relation  $E$  is  $D$ -stable if and only if  $\hat{E}$  is closed.*

The proof of this lemma is similar as the case of vector fields. For the self-contained, we state the proof.

*Proof.* Suppose that  $E$  is  $D$ -stable. Let  $\{(x_{\alpha}, y_{\alpha})\}$  be a net in  $\hat{E}$  which converges to a point  $(x, y) \in X \times X$ . Since  $y_{\alpha} \in \hat{E}(x_{\alpha}) = D(x_{\alpha})$ , by the definition of  $D(\cdot)$ , we have  $y \in D(x) = \hat{E}(x)$  and so  $(x, y) \in \hat{E}$ . This means that  $\hat{E}$  is closed. Conversely, suppose that  $\hat{E}$  is closed. By Corollary 1.3, we have  $\hat{E}$  is an equivalence relation. Fix an element  $x \in X$ . Trivially,  $\hat{E}(x) = \overline{E(x)} \subseteq D(x)$ . For  $y \in D(x)$ , we have nets  $(x_{\alpha}), (y_{\alpha})$  in  $X$  such that  $\lim x_{\alpha} = x$ ,  $\lim y_{\alpha} = y$ , and  $y_{\alpha} \in E(x_{\alpha})$ . Since  $(x_{\alpha}, y_{\alpha}) \in \hat{E}$  and  $\hat{E}$  is closed, we have that  $(x, y) \in \hat{E}$ . Hence  $y \in \hat{E}(x)$  and so  $\hat{E}(x) = D(x)$ .  $\square$

**Corollary 3.2.** *Suppose that  $X$  is Hausdorff and that  $\mathcal{F}$  is a decomposition on  $X$ . Then  $\mathcal{F}$  is  $D$ -stable if and only if  $\mathcal{F}$  is  $R$ -closed.*

Now we consider the following Lyapunov stable type condition. We call that  $E$  is  $L$ -stable if for any open neighborhood  $U$  of  $\hat{E}(x)$  and for an element  $x$  of  $X$ , there is a  $E$ -saturated open neighborhood  $V$  of  $\hat{E}(x)$  contained in  $U$ .

**Lemma 3.3.** *Assume that  $E$  is an  $L$ -stable equivalence relation on  $X$ . If  $X$  is  $T_1$ , then  $\hat{E}$  is an equivalence relation. If  $X$  is compact metrizable, then  $\hat{E}$  is a closed equivalence relation.*

*Proof.* Fix  $x \in \hat{E}(y)$ . Then every neighborhood of  $x$  meets  $E(y)$ . Since  $E$  is  $L$ -stable, every neighborhood of  $\hat{E}(x)$  contains  $E(y)$ . Because  $X$  is  $T_1$ , the intersection of the neighborhoods of a closed set is exactly the closed set itself. Hence,  $y \in E(y) \subseteq \hat{E}(x)$ . This shows that  $\hat{E}$  is symmetric. By Lemma 1.2, we have that  $\hat{E}$  is an equivalence relation. Suppose that  $X$  is  $T_4$ . To apply Theorem 3.10 [Ke] for  $\hat{E}$ , we will show that  $\tilde{U} := \cup\{x \in U \mid \hat{E}(x) \subseteq U\}$  is open for any open subset  $U$  of  $X$ . Fix  $x \in \tilde{U}$ . By the normality, there is a closed neighborhood  $V$  of  $\hat{E}(x)$  contained in  $U$ . Since  $E$  is  $L$ -stable, there is an  $E$ -saturated neighborhood  $A$  of  $\hat{E}(x)$  contained in  $V$ . Since  $V$  is closed, we have  $\hat{E}(A) \subseteq V \subseteq U$  and so  $\hat{E}(A) \subseteq \tilde{U}$ . Since  $\hat{E}(A)$  is a  $\hat{E}$ -saturated neighborhood of  $x$ , we have that  $\tilde{U}$  is open. By Theorem 3.10 [Ke], the quotient map  $q : X \rightarrow X/\hat{E}$  is closed. Since each fiber  $\hat{E}(x)$  for an element  $x$  of  $X$  is compact, we have that  $q$  is perfect. By Lemma 2.1, we obtain that  $X/\hat{E}$  is Hausdorff. By Lemma 2.2, we have that  $\hat{E}$  is closed.  $\square$

We state that the equivalence between the  $(R)$ -closedness and the  $L$ -stability.

**Proposition 3.4.** *Suppose that  $E$  is an equivalence relation on a compact metrizable space  $X$ . Then  $\hat{E}$  is a closed equivalence relation if and only if  $E$  is  $L$ -stable.*

*Proof.* By the previous lemma, it suffices to show that if  $\hat{E}$  is a closed equivalence relation, then  $E$  is  $L$ -stable. By Lemma 1.5, we have that the quotient map  $q : X \rightarrow X/\hat{E}$  is closed. For any open neighborhood  $U$  of  $\hat{E}(x)$  for each  $x \in X$ , we have  $X - U$  is closed and so  $q(X - U)$  is closed. Then  $S := q^{-1}(q(X - U)) = \hat{E}(X - U)$  is closed such that  $S \cap \hat{E}(x) = \emptyset$ , because  $q(x) \notin q(X - U)$ . Then  $X - S$  is open  $\hat{E}$ -saturated and so  $E$ -saturated such that  $\hat{E}(x) \subseteq X - S \subseteq U$ .  $\square$

This implies the following result.

**Corollary 3.5.** *Suppose that  $\mathcal{F}$  is a decomposition on a compact metrizable space. Then  $\mathcal{F}$  is  $R$ -closed if and only if  $\mathcal{F}$  is  $L$ -stable.*

We summarize the properties.

**Corollary 3.6.** *Let  $E$  be an equivalence relation on a compact metrizable space. The following are equivalent:*

- 1)  $E$  is  $R$ -closed (i.e.  $\hat{E}$  is closed).
- 2)  $E$  is  $D$ -stable.
- 3)  $E$  is  $L$ -stable.

#### 4. ON THE DENSITY OF $E$

Consider the case with dense orbits. Recall that  $X$  is a connected topological space.

**Proposition 4.1.** *Suppose that an equivalence relation  $E$  of  $X$  has a dense element. The following are equivalent:*

- 1)  $\hat{E}$  is symmetric.
- 2)  $\hat{E}$  is an equivalence relation.
- 3)  $\hat{E}$  is a closed equivalence relation.
- 4)  $E$  is minimal.

*Proof.* Trivially 3)  $\Rightarrow$  2). By Lemma 1.2, 2)  $\iff$  1). We will show that 2)  $\Rightarrow$  4). If  $\hat{E}$  is an equivalence relation, then  $X$  is decomposed into the closures of elements of  $\mathcal{F}_E$ . Since there is a dense element of  $E$ , we have  $\mathcal{F}_{\hat{E}} = \{X\}$  and so  $E$  is minimal. Finally, we will show that 4)  $\Rightarrow$  3). If  $E$  is minimal, then the closure of each element of  $\mathcal{F}_E$  is the entire  $X$  and so  $\hat{E} = X \times X$  is a closed equivalence relation trivially.  $\square$

Note that there is a recurrent flow with a dense orbit on a compact metrizable space such that  $\hat{E}_{\mathcal{F}}$  is not symmetric but transitive, where  $\mathcal{F}$  is the set of orbits of a flow (e.g. p.764 [Go]). We say that an element  $x$  of  $X$  is pointwise almost periodic if  $\hat{E}(y) = \hat{E}(x)$  for any  $y \in \hat{E}(x)$ . Considering the closure of an orbit of each  $x \in X$  as the whole topological space, we obtain the following statement.

**Corollary 4.2.** *Let  $x$  be an element of  $X$ . The following are equivalent:*

- 1)  $x$  is pointwise almost periodic.
- 2) the restriction  $\hat{E}|_{\hat{E}(x)}$  to  $\hat{E}(x)$  is symmetric
- 3)  $\hat{E}|_{\hat{E}(x)}$  is a closed equivalence relation.

## 5. CODIMENSION ONE FOLIATIONS

In this section, we consider  $\mathcal{F}$  as a codimension one foliation. Let  $M$  be a compact connected manifold and  $\mathcal{F}$  a continuous codimension one foliation on  $M$  tangent or transverse to the boundaries. Recall that  $\mathcal{F}$  is said to be compact if each leaf of  $\mathcal{F}$  is compact. Note that every codimension one compact foliation on a compact manifold has no infinite holonomy.

**Lemma 5.1.** *If  $\mathcal{F}$  is pointwise almost periodic, then  $\mathcal{F}$  is minimal or compact.*

*Proof.* Recall each minimal set is either a closed leaf, an exceptional minimal set, or the whole manifold. Since  $\mathcal{F}$  is pointwise almost periodic, any compact leaf of  $\mathcal{F}$  has no infinite holonomy. Suppose  $\mathcal{F}$  is not minimal. By the almost periodicity, each proper leaf is compact and there are no locally dense leaves. Then  $\mathcal{F}$  consists of compact leaves and exceptional leaves. We will show that there is a compact leaf  $L$  of  $\mathcal{F}$ . Otherwise,  $M$  is the union of exceptional leaves. By Theorem [S],  $M$  consists of finitely many local exceptional minimal sets. Since each exceptional minimal sets are nowhere dense, we have that  $M$  is nowhere dense. This contradicts to that  $M$  is a surface. Then the union  $C$  of compact leaves is nonempty. Since any compact leaf of  $\mathcal{F}$  has no infinite holonomy, we have that  $C$  is open. By Theorem 4.1.1(p.94)[HH], we have that  $\partial C$  contains no exceptional minimal sets and so  $C$  is clopen. Hence  $M$  consists of compact leaves.  $\square$

**Theorem 5.2.** *The following are equivalent:*

- 1)  $\mathcal{F}$  is pointwise almost periodic.
- 2)  $\mathcal{F}$  is  $R$ -closed.
- 3)  $\mathcal{F}$  is minimal or compact.

*Proof.* Since  $M$  is compact metrizable, Corollary 1.4 implies that  $2) \Rightarrow 1)$ . By Lemma 5.1, we obtain  $1) \Rightarrow 3)$ . Since  $\mathcal{F}$  has no infinite holonomy, by Proposition 2.5, we have  $3) \Rightarrow 2)$ .  $\square$

**Corollary 5.3.** *Suppose that  $\mathcal{F}$  is not minimal. The following are equivalent:*

- 1)  $\mathcal{F}$  is pointwise almost periodic.
- 2)  $\mathcal{F}$  is compact.
- 3)  $M/\mathcal{F}$  is either closed interval or a circle.
- 4)  $\mathcal{F}$  is  $R$ -closed.

*Proof.* By Theorem 5.2, we obtain  $1) \iff 2) \iff 4)$ . Taking the doubling of  $M$ , we may assume that  $M$  is closed and  $\mathcal{F}$  is transversally orientable. Lemma 2.4 implies that  $3) \Rightarrow 4)$ . Thus it suffices to show that  $2) \Rightarrow 3)$ . Suppose that 2) (and 4)) holds. By Lemma 2.3,  $M/\mathcal{F}$  is Hausdorff. Since  $\mathcal{F}$  is compact, codimension one, and transversally orientable, we have that  $\mathcal{F}$  is without holonomy and so each leaf of  $\mathcal{F}$  has a product neighborhood of it. Hence  $M/\mathcal{F}$  is a closed 1-manifold and so a circle.  $\square$

## 6. FLOWS ON COMPACT SURFACES

Let  $X$  be a compact topological space and  $G$  a topological group. Recall that a subset  $S$  of  $G$  is said to be (left) syndetic if there is a compact set  $K$  of  $G$  with  $KS = G$ . Consider a flow  $(X, G)$  (i.e. a continuous (left) action of  $G$  on  $X$ ). For a point  $x \in X$  and an open  $U$  of  $X$ , let  $N(x, U) = \{g \in G \mid gx \in U\}$ . We say that  $x$  is an pointwise almost periodic point if  $N(x, U)$  is syndetic for every neighborhood  $U$  of  $x$ . A flow  $G$  is pointwise almost periodic if every point  $x \in X$  is pointwise almost periodic. Note that a flow on a compact metrizable space is pointwise almost periodic if and only if the set of orbits is pointwise almost periodic. A point  $x \in X$  is recurrent (or Poisson stable) if  $x \in \alpha(x) \cap \omega(x)$ , where  $\alpha(x)$  (resp.  $\omega(x)$ ) is an alpha (resp. omega) limit set of  $x$ . A flow  $G$  is (pointwise) recurrent if every point of  $X$  is recurrent. Note that a pointwise almost periodic flow on a compact metrizable space  $X$  is pointwise recurrent and that an pointwise almost periodic flow on it is equivalent to a flow whose orbit closures form a decomposition of it.  $G$  is  $R$ -closed if  $R := \{(x, y) \mid y \in \overline{G(x)}\}$  is closed. Notice that an  $R$ -closed flow on a compact metrizable space  $X$  is pointwise almost periodic.

Let  $M$  be a compact connected surface and  $w$  a continuous vector fields of  $M$ . Note that if  $w$  is pointwise recurrent, then  $w$  is tangent to the boundary  $\partial M$ . In this section, we consider  $\mathcal{F}$  as the set of the orbits of the vector field  $w$ . Write  $M/w := M/\mathcal{F}$  and  $M/\overline{w} := M/\tilde{\mathcal{F}}$ . Now  $R = \hat{E}_{\mathcal{F}} = \{(x, y) \mid y \in \overline{O_w(x)}\}$ .

**Lemma 6.1.** *If  $w$  is not minimal but pointwise recurrent, then  $M = \text{Sing}(w) \sqcup \text{Per}(w)$ .*

*Proof.* Since  $w$  is recurrent, we have that  $w$  preserves the boundary of  $M$ , has no saddles, and is nonwandering (i.e.  $x \in J^+(x)$  for any  $x \in M$  where  $J^+(x) = \{y \in M \mid t_n x_n \rightarrow y \text{ for some } x_n \rightarrow x \text{ and } t_n \rightarrow +\infty\}$ ). By taking the double of  $M$ , we may assume that  $M$  is closed. Since the Denjoy flow has wandering points, by Corollary 3.5 [A], we have  $M = \text{Sing}(w) \sqcup \text{Per}(w)$ .  $\square$

This implies the following corollary.

**Corollary 6.2.** *Let  $w$  be a continuous vector fields of a compact connected surface  $M$ . The following are equivalent:*

- 1)  $w$  is pointwise recurrent.
- 2)  $w$  is pointwise almost periodic.
- 3)  $w$  is either minimal or pointwise periodic.

Recall that a vector field is called nontrivial if it is neither identical nor minimal.

**Corollary 6.3.** *Suppose that  $w$  is nontrivial. Then  $w$  is pointwise recurrent if and only if  $w$  is pointwise periodic.*

Next, we show the openness of  $\text{Per}(w)$ .

**Lemma 6.4.** *Suppose that  $w$  is pointwise recurrent. Then  $\text{Per}(w)$  is open. Moreover, if  $\text{Per}(w)$  is nonempty, then each connected component of  $\text{Per}(w)$  is an annulus or a torus, and  $\text{Per}(w)/w|_{\text{Per}(w)}$  is a 1-manifold.*

*Proof.* By Lemma 6.1, we have  $M = \text{Sing}(w) \sqcup \text{Per}(w)$ . Suppose that there are periodic points. By Flow box theorem, each periodic orbits has a product neighborhood of it which consists of periodic orbits. Hence the set  $\text{Per}(w)$  of periodic orbits of  $w$  is open and the quotient space of  $\text{Per}(w)$  is a 1-manifold. Then each connected component of  $\text{Per}(w)$  is an annulus or a torus.  $\square$

From now on, we assume that  $M$  is orientable. Note that we can obtain the similar results for the non-orientable case, by taking the double cover of  $X$ . For simplicity, we consider only the orientable case. Recall that  $w$  is  $R$ -closed if  $R := \{(x, y) \in M \times M \mid y \in \overline{O(x)}\}$  is closed.

**Lemma 6.5.** *If  $w$  is nontrivial and  $R$ -closed, then  $M/w$  is an closed interval or a circle, and each connected component of  $\partial\text{Per}(w) := \overline{\text{Per}(w)} - \text{Per}(w)$  is an element of  $\mathcal{F}$  which is a center.*

*Proof.* By Lemma 2.3, we have that  $M/\overline{w} = M/w$  is Hausdorff. By Lemma 6.4, we have that  $M/w$  is an closed interval or a circle and that each connected component of  $\partial\text{Per}(w)$  is an element of  $\mathcal{F}$  which is a center.  $\square$

Now, we state the characterization of  $R$ -closedness.

**Theorem 6.6.** *Suppose that  $w$  is nontrivial. The following are equivalent:*

- 1)  $w$  is  $R$ -closed.
- 2)  $w$  consists of at most two centers and other periodic orbits.
- 3) The orbit space  $M/w$  is either a circle reduced from a torus or is a closed interval reduced from a closed disk, a closed annulus or a sphere.

*In particular,  $w$  is pointwise periodic if one of equivalent conditions holds.*

*Proof.* By Lemma 6.5, we have that 1)  $\Rightarrow$  2) and that 1)  $\Rightarrow$  3). Suppose that 3) holds. Since  $M/w$  is Hausdorff, we obtain that  $M/w = M/\overline{w}$  is Hausdorff. By Lemma 2.3, we have that  $w$  is  $R$ -closed. Suppose that 2) holds. By Flow box theorem, we have that  $\text{Per}(w)$  is open and  $\text{Per}(w)/w|_{\text{Per}(w)} = \text{Per}(w)/\overline{w}|_{\text{Per}(w)}$  is a 1-manifold. Since singularities are finite and so isolated,  $M/w = M/\overline{w}$  is Hausdorff. By Lemma 2.3, we have that  $w$  is  $R$ -closed.  $\square$

Recall that  $R$ -closedness and  $D$ -stability for vector fields are equivalent. Therefore Theorem 4.3, 4.4 [APS] are related to our characterization. Consider the regular case. Recall that we say that a vector field  $w$  is regular if  $w$  is topologically



equivalent to a vector field whose singular points are non-degenerate. In the two dimensional case, each singularity of a regular vector field is either a sink, a source, a (topological) saddle, or a center.

**Lemma 6.7.** *Suppose  $w$  is nontrivial. Then  $R$  is closed if and only if  $w$  is pointwise almost periodic and topologically equivalent to a regular vector field.*

*Proof.* Suppose that  $R$  is closed. By Corollary 6.2, we have that  $w$  is pointwise periodic. Hence  $w$  is topologically equivalent to a regular vector field. Conversely, suppose that  $w$  is pointwise almost periodic and regular. By Corollary 6.2,  $w$  is pointwise periodic. Since  $w$  has no saddle points, by the Poincaré-Hopf index formula, we obtain that  $w$  consists of periodic orbits and at most two centers. By Theorem 6.6, we obtain  $w$  is  $R$ -closed.  $\square$

Now we state the following characterization in the regular case.

**Proposition 6.8.** *Let  $w$  be a continuous regular nontrivial vector field of  $M$ . The following are equivalent:*

- 1)  $w$  is pointwise recurrent.
- 2)  $w$  is pointwise almost periodic.
- 3)  $w$  is  $R$ -closed.
- 4)  $w$  is pointwise periodic.
- 5)  $\text{Sing}(w)$  are centers and  $w$  has neither exceptional minimal sets nor limit cycles.
- 6) the orbit space  $M/w$  is either a circle on a torus, or a closed interval on either a closed annulus, a closed disk, or a sphere.

*Proof.* By Corollary 6.2 and Lemma 6.7, 1)  $\Leftrightarrow$  2)  $\Leftrightarrow$  3)  $\Leftrightarrow$  4). By Theorem 6.6, 3)  $\Leftrightarrow$  6). Suppose that 4) holds. Since  $w$  is regular,  $\text{Sing}(w)$  are centers and so 5) holds. Finally, we will show that 5)  $\Rightarrow$  4). Suppose that 5) holds. We will show that there are no exceptional orbits. Otherwise there is a local exceptional minimal set  $K$ . Then there is a periodic orbit which is contained in the closure of  $K$ , because  $\text{Sing}(w)$  are centers and  $w$  has no exceptional minimal sets. So this means that there is a limit cycle, which contradicts. We will show that each orbits are closed. Otherwise the boundary of the set of non-closed orbits contains minimal sets. Since there are no exceptional orbits and since  $\text{Sing}(w)$  are centers, the minimal sets are closed orbits and so there is limits cycles, which contradicts. Therefore 4) holds.  $\square$

By the existence of the Denjoy flow, the condition that  $w$  has no exceptional minimal sets in 4), is necessary. Finally we consider the divergence-free case. Recall that a  $C^1$  vector field  $w$  on a Riemannian manifold with a volume form  $dvol$  is divergence-free if  $L_w dvol = (\text{div } w)dvol$ , where  $L_w$  is the Lie derivative along  $w$ .

**Corollary 6.9.** *Suppose that  $w$  is regular and divergence-free and has singularities. The following are equivalent:*

- 1)  $w$  is pointwise recurrent.
- 2)  $w$  is pointwise almost periodic.
- 3)  $w$  is  $R$ -closed.
- 4)  $w$  is pointwise periodic.
- 5)  $w$  has neither saddles nor  $\partial$ -saddles.
- 6)  $w$  has neither saddles nor  $\partial$ -saddles and is structurally stable in the set of divergence-free vector fields.
- 7)  $M/w$  is a closed interval.

If an equivalent condition holds, then  $M$  is either a closed disk or a sphere and has one or two singularities, which are centers.

*Proof.* By proposition 6.8, we obtain  $1) \Leftrightarrow 2) \Leftrightarrow 3) \Leftrightarrow 4) \Leftrightarrow 7)$ . By taking the doubling of  $M$  if necessary, we may assume that  $M$  has no boundaries. Obviously,  $4) \Rightarrow 5)$ . We will show that  $5) \Rightarrow 6)$ . Suppose 5) holds. Since  $w$  is regular and has singularities, by Poincaré-Hopf theorem, we have  $M = S^2$ . By Theorem 2.1.2[MW],  $w$  is structurally stable in the set of divergence-free vector fields. Finally, we will show that  $6) \Rightarrow 7)$ . Suppose 6) holds. By Theorem 1.4.6.[MW],  $w$  is determined by the saddle connection diagram (the union of saddle connection) up to topological equivalence. Since  $w$  has no saddles, we have  $w$  has two centers and others periodic orbits. By proposition 6.8, we have that 7) holds.  $\square$

## 7. EXAMPLES

Recall a well-known fact that a point in a compact metrizable space is pointwise almost periodic if and only if its orbit closure is a minimal set. On the other hand, this is not true for general cases (see following two examples).

**Example 1.** Let  $G$  be a trivial group and  $X = \{0, 1\}$  a two point set with a topology  $\{\emptyset, X, \{1\}\}$ . Since the closed sets are  $\emptyset, X, \{0\}$ , the closed sets of the product space are  $\emptyset, \{(0, 0)\}, \{0\} \times X, X \times \{0\}, X \times X - \{(1, 1)\}, X \times X$ . Trivially  $X$  is compact and  $G$  is periodic. In particular,  $G$  is pointwise almost periodic as a flow.  $R = \{(0, 0), (1, 1), (1, 0)\} = X \times X - \{(0, 1)\}$  is not closed. Since  $\overline{G1} = X$ , the  $X/\tilde{G} = X$  is not Hausdorff. Moreover the orbit of 1 is not minimal but periodic.

A next example shows them even if  $X$  is a  $T_1$ -space

**Example 2.** Let  $\alpha \in \mathbb{R} - \mathbb{Q}$  be an irrational number,  $G = \mathbb{Z}$  a discrete group and  $X = \mathbb{R}/\mathbb{Z}$  a topological space with a topology  $\tau = \{\emptyset\} \sqcup \{U, U \setminus F \mid U \text{ is cofinite in } X\}$ , where  $F := \mathbb{Z}\alpha/\mathbb{Z} \subset X$ .  $G$  acts by  $n \cdot [x] := [\alpha n + x]$ . Then  $X$  is a compact  $T_1$ -space,  $G$  is continuous and pointwise almost periodic as a flow,  $R$  is not closed, and  $\overline{Gx}$  for any  $x \in X - F$  is not minimal.

Indeed, notice that the set  $C_X$  of closed subsets are  $\{\emptyset, X\} \sqcup \{E, E \cup F \mid E \text{ is finite in } X\}$ . For any  $n \in G$ , since  $n \cdot F = F$ , we have that  $n \cdot C_X = C_X$  and so  $G$  acts continuously. For any  $x \in F$ , each neighborhood of  $x$  is cofinite. Hence  $X$  is compact. For any  $x \in X$ , every neighborhood of  $x$  contain all but finitely many points of its orbit and so  $G$  is pointwise almost periodic as a flow. Since  $\tau$  contains the cofinite topology, each point of  $X$  is closed and so  $X$  is  $T_1$ . For any  $x \in X - F$ , since  $\overline{Gx} = X$  and  $F$  is a proper closed subset, we obtain that  $\overline{Gx} = M$  is not minimal. For any  $x, y \in X - F$ , we have  $Gx \times \{y\} \subset R$ . Since any neighborhood of 0 is cofinite and so meets  $Gx$ , we obtain  $0 \in \overline{Gx}$  and so  $(0, y) \in \overline{Gx \times \{y\}} \subseteq \overline{R}$ . Since  $y \notin F = \overline{G0}$ , we have  $(0, y) \notin R$ . Thus  $R$  is not closed.

A next example also shows that the Lemma 2.3 is not true in general.

**Example 3.** Let  $\alpha \in \mathbb{R} - \mathbb{Q}$  be an irrational number,  $X = \mathbb{R}/\mathbb{Z}$ ,  $F = \{\alpha n \mid n \in \mathbb{Z}\}/\mathbb{Z}$ ,  $\tau$  the topology on  $X$  induced by the Euclidean topology on  $\mathbb{R}$ , and  $E := 1_X \cup (F \times F)$  an equivalence relation on  $X$ . Equip with  $X$  a topology generated by  $\tau$  and  $\{U \setminus F \mid U \in \tau\}$ . Put  $\mathcal{F} = \mathcal{F}_E$ . Then  $X$  is a non-compact Hausdorff space,  $\mathcal{F}$  is pointwise almost periodic,  $R = \hat{E}$  is closed, and  $X/\tilde{\mathcal{F}}$  is non-Hausdorff. Indeed, since  $F$  is closed, we have  $\hat{E} = E$  and so  $\mathcal{F}$  is pointwise almost periodic.

Now we will show that  $R$  is closed. Fix  $x \neq y \in X$ . If  $x, y \notin F$ , then there are disjoint neighborhoods  $U_x, U_y \in \tau$  of  $x$  and  $y$  on  $X$ . Then  $(U_x \setminus F) \times (U_y \setminus F)$  is a neighborhood of  $(x, y)$  which does not meet  $\hat{E}$ . If  $x \in F$  and  $y \notin F$ , then there are disjoint neighborhoods  $U_x, U_y \in \tau$  of  $x$  and  $y$  on  $X$ . Then  $U_x \times (U_y \setminus F)$  is a neighborhood of  $(x, y)$  which does not meet  $\hat{E}$ . This shows that  $\hat{E}$  is closed. For  $x \in X - F$ , since  $x$  and  $F$  can't be separated by disjoint open sets,  $X/\tilde{F}$  is non-Hausdorff.

A next example also shows that there is a topological dynamics on a compact surface such that  $G$  is pointwise almost periodic and  $R$  is not closed but symmetric.

**Example 4.** Let  $G$  be an additive group  $\mathbb{Z}$  or  $\mathbb{R}$ ,  $X = \{(x, y) \mid x^2 + y^2 \leq 1\}$  a unit closed disk, and  $v = (y(1 - (x^2 + y^2)), -x(1 - (x^2 + y^2)))$  a vector field.  $G$  acts by  $t \cdot (x, y) := v_t(x, y)$ . Then the fix point set of  $G$  is the union of the boundary and the origin, the other orbits are periodic orbits, and the orbit space is not  $T_1$ . Hence  $G$  is pointwise almost periodic and  $R$  is not closed but symmetric.

Indeed, the boundary points are singular and is not separated by saturated neighborhoods. Let  $S_r := \{(x, y) \mid x^2 + y^2 = r^2\}$  be a circle. Then  $G$  acts  $S_r$  as rotations. Hence  $G$  is pointwise almost periodic. For  $r_n := 1 - 1/\sqrt{\pi n}$ ,  $\mathbb{Z} \subseteq G$  acts  $S_{r_n}$  as irrational rotations. Since  $(r_n, -r_n) \in R$  but  $(1, -1) \notin R$ , we have  $R$  is a non-closed equivalent relation.

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